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International Journal of Solids and Structures 43 (2006) 4082–4096

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

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Thermal buckling of imperfect functionally graded plates

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Received 5 April 2005
Available online 21 June 2005

Abstract

Thermal buckling analysis of rectangular functionally graded plates (FGPs) with geometrical imperfections is presented in this paper. The equilibrium, stability, and compatibility equations of an imperfect functionally graded plate are derived using the classical plate theory. It is assumed that the nonhomogeneous mechanical properties of the plate, graded through thickness, are described by a power function of the thickness variable. The plate is assumed to be under three types of thermal loading as uniform temperature rise, nonlinear temperature rise through the thickness, and axial temperature rise. Resulting equations are employed to obtain the closed-form solutions for the critical buckling temperature change of an imperfect FGP. The results are reduced and compared with the results of perfect functionally graded and imperfect isotropic plates.

Keywords: Thermal buckling; Functionally graded plate; Imperfection

1. Introduction

A comprehensive work on the buckling of structures is presented by Brush and Almroth (1975). They have examined the effect of initial imperfections on the critical loads. A review of research on thermal buckling of plates and shells since the first work in 1950s is presented by Thornton (1993). He has described the elastic thermal buckling of metallic as well as composite plates and shells. Turvey and Marshall (1995) studied buckling and postbuckling of composite plates due to mechanical and thermal loads.

The initial geometric imperfections are inherent in many real structures. Therefore, many investigations are conducted on the stability analysis of imperfect structures. Elastic, plastic, and creep buckling of imperfect cylinders under mechanical and thermal loads is studied by Eslami and Shariyat (1997). Mossavarali et al. studied the thermoelastic buckling of isotropic and orthotropic plates with imperfections (Mossavarali

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0020-7683/$ - see front matter
doi:10.1016/j.ijsolstr.2005.04.005

Nomenclature

a, b	plate length and width
$E(z)$, E_c, E_m	elasticity modulus of FGM, ceramic and metal
h	plate thickness
k	power law index
k_i	curvatures
$K(z)$, K_c, K_m	thermal conductivity of FGM, ceramic and metal
m, n	number of half waves in x- and y-directions
N_i, M_i	force and moment resultants
T	temperature
u, v, w	displacement components
w^*	initial imperfection
x, y, z	rectangular Cartesian coordinates
$\alpha(z)$, α_c, α_m	coefficient of thermal expansion of FGM, ceramic and metal
γ_{xy}	shear strain
ϵ_x, ϵ_y	normal strains
μ	imperfection size
v_0	Poisson's ratio
ΔT_{cr}	critical buckling temperature change

et al., 2000; Mossavarali and Eslami, 2002). Murphy and Ferreira (2001) investigated thermal buckling analysis of clamped rectangular plates based on the energy consideration. They determined the ratio of the critical temperature for a perfect flat plate to the one for an imperfect plate as a function of the initial imperfection size. The study includes experimental results. Eslami and Shahsiah (2001) reported thermal buckling of imperfect circular cylindrical shells based on the Wan-Donnell and Koiter imperfection models.

The development of new materials with new constitutive models have necessitated more research in the area of stability analysis. Functionally graded materials (FGMs) are of these new and high-temperature resistant materials in which material constitution vary continuously across the thickness of a structure. Some works about the stability of FGM structures are introduced in the following. Javaheri and Eslami (2001, 2002a,b,c) reported mechanical and thermal buckling of rectangular functionally graded plates. They used energy method and mainly reached to the closed-form solutions. Najafizadeh and Eslami (2002a,b) studied thermoelastic stability of circular FGPs. The research on thermal buckling of functionally graded cylindrical shells is introduced by Shahsiah and Eslami (2003a,b). Ma and Wang (2004) employed the third order shear deformation plate theory to solve the axisymmetric bending and buckling problems of functionally graded circular plates. Shen (2004) represented thermal postbuckling behavior of functionally graded cylindrical shells with temperature dependent properties. He considered initial geometric imperfections in the analysis. Three dimensional thermal buckling analysis of functionally graded materials, using finite element method, is reported by Na and Kim (2004).

In the present article, the influence of geometrical imperfections on thermal instability of FGPs is investigated. The plate is graded through the thickness direction according to a power law function. The classical plate theory is used with a double-sine function for the geometric imperfection along the x- and y-directions. The boundary conditions along the four edges of the plate are assumed to be fixed-simply supported. The buckling of the plate under three types of thermal loads are obtained. The thermal loads are assumed to be the uniform temperature rise, nonlinear thermal gradient through the thickness direction, and the axial temperature variation along the x-direction. Closed formed solutions are obtained and given for all three types of the assumed thermal loads. The results are validated with the results of homogeneous plates under the same types of thermal loads

2. Functionally graded plates

Functionally graded materials (FGMs) are microscopically inhomogeneous materials in which the mechanical properties vary smoothly and continuously through the thickness. This is achieved by gradually changing the volume fraction of the constituent materials. These materials are typically made from a mixture of ceramic and metal or a combination of different metals (Reddy and Chin, 1998). The ceramic part provides high-temperature resistance and the metal part prevents fracture due to thermal loadings. We assume that the modulus of elasticity E, the coefficient of thermal expansion α, and conductivity K change in the thickness direction z, while the Poisson's ratio v is assumed to be constant. The material properties of FGP are introduced as (Javaheri and Eslami, 2002a)

$$\begin{aligned} E(z) &= E_{\mathrm{m}} + E_{\mathrm{cm}}\left(\frac{2z+h}{2h}\right)^k, \\ \alpha(z) &= \alpha_{\mathrm{m}} + \alpha_{\mathrm{cm}}\left(\frac{2z+h}{2h}\right)^k, \\ K(z) &= K_{\mathrm{m}} + K_{\mathrm{cm}}\left(\frac{2z+h}{2h}\right)^k, \\ v(z) &= v_0, \end{aligned} \tag{1}$$

where

$$\begin{aligned} E_{\mathrm{cm}} &= E_{\mathrm{c}} - E_{\mathrm{m}}, \\ \alpha_{\mathrm{cm}} &= \alpha_{\mathrm{c}} - \alpha_{\mathrm{m}}, \\ K_{\mathrm{cm}} &= K_{\mathrm{c}} - K_{\mathrm{m}}, \end{aligned} \tag{2}$$

and subscripts 'm' and 'c' refer to the metal and ceramic constituents, respectively. The variable z is the thickness coordinate $(-h/2 \leqslant z \leqslant h/2)$, where h is the thickness of the plate and k is the power law index which takes values greater than or equal to zero. The variation of the composition of ceramic and metal is linear for $k = 1$. The value of k equal to zero represents a fully ceramic plate.

3. Analysis

Consider a rectangular thin flat plate of length a width b and thickness h made of FGM. The plate is subjected to thermal loading. Rectangular Cartesian coordinates (x, y, z) are assumed for derivations.

The constitutive relations are written as (Javaheri and Eslami, 2002a)

$$\begin{aligned} N_x &= \frac{E_1}{1-v_0^2}(\epsilon_x + v_0\epsilon_y) + \frac{E_2}{1-v_0^2}(k_x + v_0 k_y) - \frac{\Phi_{\mathrm{m}}}{1-v_0}, \\ N_y &= \frac{E_1}{1-v_0^2}(\epsilon_y + v_0\epsilon_x) + \frac{E_2}{1-v_0^2}(k_y + v_0 k_x) - \frac{\Phi_{\mathrm{m}}}{1-v_0}, \\ N_{xy} &= \frac{E_1}{2(1+v_0)}\gamma_{xy} + \frac{E_2}{1+v_0}k_{xy}, \end{aligned}$$

$$M_x = \frac{E_2}{1-v_0^2}(\epsilon_x + v_0\epsilon_y) + \frac{E_3}{1-v_0^2}(k_x + v_0 k_y) - \frac{\Phi_b}{1-v_0},$$
$$M_y = \frac{E_2}{1-v_0^2}(\epsilon_y + v_0\epsilon_x) + \frac{E_3}{1-v_0^2}(k_y + v_0 k_x) - \frac{\Phi_b}{1-v_0},$$
$$M_{xy} = \frac{E_2}{2(1+v_0)}\gamma_{xy} + \frac{E_3}{1+v_0}k_{xy} \quad (3)$$

where

$$E_1 = E_m h + \frac{E_{cm} h}{k+1},$$
$$E_2 = E_{cm} h^2 \left(\frac{1}{k+2} - \frac{1}{2k+2}\right),$$
$$E_3 = \frac{E_m h^3}{12} + E_{cm} h^3 \left(\frac{1}{k+3} - \frac{1}{k+2} + \frac{1}{4(k+1)}\right),$$
$$\Phi_m = \int_{\frac{-h}{2}}^{\frac{h}{2}} \left[E_m + E_{cm}\left(\frac{2z+h}{2h}\right)^k\right]\left[\alpha_m + \alpha_{cm}\left(\frac{2z+h}{2h}\right)^k\right]\Delta T(x,y,z)\,dz,$$
$$\Phi_b = \int_{\frac{-h}{2}}^{\frac{h}{2}} \left[E_m + E_{cm}\left(\frac{2z+h}{2h}\right)^k\right]\left[\alpha_m + \alpha_{cm}\left(\frac{2z+h}{2h}\right)^k\right]\Delta T(x,y,z) z\,dz. \quad (4)$$

In the above equations, N_i and M_i are force and moment resultants, respectively, ϵ_x and ϵ_y are the normal strains and γ_{xy} is shear strain at the middle surface of the plate. The curvatures are shown by k_i. Strains and curvatures are related to the displacement components as (Brush and Almroth, 1975)

$$\epsilon_x = u_{,x} + \tfrac{1}{2}w_{,x}^2,$$
$$\epsilon_y = v_{,y} + \tfrac{1}{2}w_{,y}^2,$$
$$\gamma_{xy} = u_{,y} + v_{,x} + w_{,x}w_{,y},$$
$$k_x = -w_{,xx},$$
$$k_y = -w_{,yy},$$
$$k_{xy} = -w_{,xy}, \quad (5)$$

where u, v, and w are displacement components along the x-, y-, and z-directions, respectively. The equilibrium equations of a perfect functionally graded plate are (Javaheri and Eslami, 2002a)

$$N_{x,x} + N_{xy,y} = 0,$$
$$N_{xy,x} + N_{y,y} = 0,$$
$$M_{x,xx} + 2M_{xy,xy} + M_{y,yy} + N_x w_{,xx} + 2N_{xy} w_{,xy} + N_y w_{,yy} = 0. \quad (6)$$

Considering Eqs. (3) and (5), the equilibrium equations may be described in more usable form as

$$N_{x,x} + N_{xy,y} = 0,$$
$$N_{xy,x} + N_{y,y} = 0,$$
$$D\nabla^4 w - N_x w_{,xx} - 2N_{xy} w_{,xy} - N_y w_{,yy} - \frac{E_2}{(1-v_0)E_1}(\Phi_{m,xx} + \Phi_{m,yy}) + \frac{1}{1-v_0}(\Phi_{b,xx} + \Phi_{b,yy}) = 0, \quad (7)$$

where

$$D = \frac{E_1 E_3 - E_2^2}{(1-v_0^2)E_1}. \quad (8)$$

For a slightly imperfect plate, let $w^*(x, y)$ denotes a known small imperfection. This parameter represents a small deviation of the plate middle plane from a flat shape. To consider imperfections, the displacement component w in the third relation of Eq. (7) must be replaced by $(w + w^*)$. Note that the term $D\nabla^4 w$ will be unchanged, because this term is obtained from the expressions for bending moments and these moments depend not on the total curvature but only on the change in curvature of the plate (Timoshenko and Gere, 1961). Therefore, equilibrium equation (7) are modified as

$$\begin{aligned}
&N_{x,x} + N_{xy,y} = 0,\\
&N_{xy,x} + N_{y,y} = 0,\\
&D\nabla^4 w - N_x(w_{,xx} + w^*_{,xx}) - 2N_{xy}(w_{,xy} + w^*_{,xy}) - N_y(w_{,yy} + w^*_{,yy})\\
&\quad - \frac{E_2}{(1-\nu_0)E_1}(\Phi_{\mathrm{m},xx} + \Phi_{\mathrm{m},yy}) + \frac{1}{1-\nu_0}(\Phi_{\mathrm{b},xx} + \Phi_{\mathrm{b},yy}) = 0.
\end{aligned} \tag{9}$$

The stability equations of the plate may be derived by the adjacent equilibrium criterion (Brush and Almroth, 1975). Assume that the equilibrium state of a FGP under thermal load is defined in terms of displacement components u_0, v_0, and w_0. The displacement components of a neighboring stable state differ by u_1, v_1, and w_1 with respect to the equilibrium position. Thus, the total displacements of a neighboring state are

$$\begin{aligned}
&u = u_0 + u_1,\\
&v = v_0 + v_1,\\
&w = w_0 + w_1.
\end{aligned} \tag{10}$$

Similarly, the force resultants of a neighboring state may be related to the state of equilibrium as

$$\begin{aligned}
&N_x = N_{x0} + \Delta N_x,\\
&N_y = N_{y0} + \Delta N_y,\\
&N_{xy} = N_{xy0} + \Delta N_{xy},
\end{aligned} \tag{11}$$

where ΔN_x, ΔN_y, and ΔN_{xy} are the increments corresponding to u_1, v_1, and w_1, respectively. Now, let N_{x1}, N_{y1}, and N_{xy1} represent the parts of ΔN_x, ΔN_y, and ΔN_{xy} that are linear in u_1, v_1, and w_1. The above relations with consideration of linear force increments are written as

$$\begin{aligned}
&N_x = N_{x0} + N_{x1},\\
&N_y = N_{y0} + N_{y1},\\
&N_{xy} = N_{xy0} + N_{xy1}.
\end{aligned} \tag{12}$$

The stability equations may be obtained by substituting Eqs. (10) and (12) in Eq. (9). We assume that the temperature variation is at last linear with respect to x- and y-directions. Upon substitution, the terms in the resulting equations with subscript 0 satisfy the equilibrium condition and therefore drop out of the equations. Also, the nonlinear terms with subscript 1 are ignored because they are small compared to the linear terms. The remaining terms form the stability equations of an imperfect rectangular FGP under thermal load as

$$\begin{aligned}
&N_{x1,x} + N_{xy1,y} = 0,\\
&N_{xy1,x} + N_{y1,y} = 0,\\
&D\nabla^4 w_1 - N_{x0}w_{1,xx} - 2N_{xy0}w_{1,xy} - N_{y0}w_{1,yy}\\
&\quad - N_{x1}(w_{0,xx} + w^*_{,xx}) - 2N_{xy1}(w_{0,xy} + w^*_{,xy}) - N_{y1}(w_{0,yy} + w^*_{,yy}) = 0.
\end{aligned} \tag{13}$$

The subscript '1' refers to the state of stability and the subscript '0' refers to the state of equilibrium conditions. Considering the first two of Eq. (13), a stress function f may be defined as

$$\begin{aligned} N_{x1} &= f_{,yy}, \\ N_{y1} &= f_{,xx}, \\ N_{xy1} &= -f_{,xy}. \end{aligned} \tag{14}$$

Substitution of Eq. (14) in the third of Eq. (13) leads to

$$D\nabla^4 w_1 - N_{x0}w_{1,xx} - 2N_{xy0}w_{1,xy} - N_{y0}w_{1,yy} - f_{,yy}(w_{0,xx} + w^*_{,xx}) + 2f_{,xy}(w_{0,xy} + w^*_{,xy}) - f_{,xx}(w_{0,yy} + w^*_{,yy}) = 0. \tag{15}$$

This equation is the stability equation for an imperfect FGP. The equation includes two dependent unknowns, w_1 and f. To obtain a second equation relating these two unknowns, the compatibility equation may be used.

Assume ϵ_{x1}, ϵ_{y1}, and γ_{xy1} denote parts of the strain components which are linear in u_1, v_1, and w_1. These strains may be written in terms of the displacement components, using Eqs. (3), (5) and (10)–(12) with consideration of the imperfection term w^*, as

$$\begin{aligned} \epsilon_{x1} &= u_{1,x} + (w_{0,x} + w^*_{,x})w_{1,x}, \\ \epsilon_{y1} &= v_{1,y} + (w_{0,y} + w^*_{,y})w_{1,y}, \\ \gamma_{xy1} &= u_{1,y} + v_{1,x} + (w_{0,x} + w^*_{,x})w_{1,y} + (w_{0,y} + w^*_{,y})w_{1,x}. \end{aligned} \tag{16}$$

Using Eq. (16), the geometrical compatibility equation is written as

$$\epsilon_{x1,yy} + \epsilon_{y1,xx} - \gamma_{xy1,xy} = 2(w_{0,xy} + w^*_{,xy})w_{1,xy} - (w_{0,xx} + w^*_{,xx})w_{1,yy} - (w_{0,yy} + w^*_{,yy})w_{1,xx}. \tag{17}$$

From the constitutive relations (3), one can write

$$\begin{aligned} \epsilon_{x1} &= \frac{1}{E_1}(N_{x1} - v_0 N_{y1} - E_2 k_{x1} + \Phi_{m1}), \\ \epsilon_{y1} &= \frac{1}{E_1}(N_{y1} - v_0 N_{x1} - E_2 k_{y1} + \Phi_{m1}), \\ \gamma_{xy1} &= \frac{2}{E_1}((1 + v_0)N_{xy1} - E_2 k_{xy1}). \end{aligned} \tag{18}$$

Substituting the above equations in Eq. (17), with the aid of Eqs. (5) and (14), leads to the compatibility equation of an imperfect FGP as

$$\frac{1}{E_1}\nabla^4 f - 2w_{1,xy}(w_{0,xy} + w^*_{,xy}) + w_{1,xx}(w_{0,yy} + w^*_{,yy}) + w_{1,yy}(w_{0,xx} + w^*_{,xx}) = 0. \tag{19}$$

Eqs. (9), (15) and (19) are the basic equations used to obtain the critical buckling load of an imperfect functionally graded plate.

3.1. Buckling of imperfect functionally graded plates under uniform temperature rise

Consider a rectangular imperfect plate made of functionally graded material which is fixed simply supported and subjected to uniform temperature rise (Meyers and Hyer, 1991). The edge conditions are defined as

$$\begin{aligned} u = v = w = M_x = 0 &\quad \text{on } x = 0, a, \\ u = v = w = M_y = 0 &\quad \text{on } y = 0, b. \end{aligned} \tag{20}$$

The plate initial temperature is assumed to be T_i. The temperature is uniformly raised to a final value T_f in which the plate buckles. The temperature change is $\Delta T = T_f - T_i$. The prebuckling resultant forces are defined as (Javaheri and Eslami, 2002a)

$$
\begin{aligned}
N_{x0} &= -\frac{\Phi_m}{1-\nu_0}, \\
N_{y0} &= -\frac{\Phi_m}{1-\nu_o}, \\
N_{xy0} &= 0,
\end{aligned}
\tag{21}
$$

where Φ_m is obtained from Eq. (4). The imperfections of the plate, considering the boundary conditions, are assumed as (Brush and Almroth, 1975; Timoshenko and Gere, 1961)

$$
w^* = \mu h \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \quad m,n = 1,2,\ldots, \tag{22}
$$

where the coefficient μ varies between 0 and 1 and μh represents the imperfection size. Also, m and n are number of half waves in x- and y-directions, respectively. The following approximate solution is seen to satisfy both the equilibrium equation (9) and the boundary conditions expressed by Eq. (20)

$$
w_0 = A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \quad m,n = 1,2,\ldots \tag{23}
$$

where A_{mn} is a constant coefficient. Substituting Eqs. (21)–(23) into the third equilibrium equation (9), the constant A_{mn} is obtained and the final approximate solution can be written as

$$
w_0 = \frac{\frac{\Phi_m}{1-\nu_0}\mu h}{D\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right] - \frac{\Phi_m}{1-\nu_0}}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}. \tag{24}
$$

Substituting prebuckling resultant forces from Eq. (21) and deflection w_0 from Eq. (23) into Eqs. (15) and (19) yield

$$
\begin{aligned}
&D\nabla^4 w_1 + \frac{\Phi_m}{1-\nu_0}(w_{1,xx} + w_{1,yy}) + \left(f_{,yy}\left(\frac{m\pi}{a}\right)^2 + f_{,xx}\left(\frac{n\pi}{b}\right)^2\right)(A_{mn} + \mu h)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \\
&\quad + 2f_{,xy}\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)(A_{mn} + \mu h)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} = 0, \\
&\frac{1}{E_1}\nabla^4 f - 2\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)(A_{mn} + \mu h)w_{1,xy}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \\
&\quad - (A_{mn} + \mu h)\left(w_{1,yy}\left(\frac{m\pi}{a}\right)^2 + w_{1,xx}\left(\frac{n\pi}{b}\right)^2\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} = 0.
\end{aligned}
\tag{25}
$$

To solve the system of Eq. (25), with the consideration of the boundary conditions (20), the approximate solutions may be considered as

$$
\begin{aligned}
w_1 &= E_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \\
f &= F_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b},
\end{aligned}
\tag{26}
$$

where E_{mn} and F_{mn} are constant coefficients that depend on m and n. Substituting approximate solutions (26) into Eq. (25) gives

$$
\begin{aligned}
R_1 = {} & E_{mn}\left\{D\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^2-\frac{\Phi_{\mathrm{m}}}{1-\nu_0}\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]\right\}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \\
& -2F_{mn}(A_{mn}+\mu h)\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2\left(\sin^2\frac{m\pi x}{a}\sin^2\frac{n\pi y}{b}-\cos^2\frac{m\pi x}{a}\cos^2\frac{n\pi y}{b}\right), \\
R_2 = {} & \frac{1}{E_1}F_{mn}\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^2\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \\
& +2E_{mn}(A_{mn}+\mu h)\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2\left(\sin^2\frac{m\pi x}{a}\sin^2\frac{n\pi y}{b}-\cos^2\frac{m\pi x}{a}\cos^2\frac{n\pi y}{b}\right),
\end{aligned}
\tag{27}
$$

where R_1 and R_2 are the residues of Eq. (25). According to the Galerkin's method, R_1 and R_2 are made orthogonal with respect to the approximate solutions as

$$
\begin{aligned}
&\int_0^a\int_0^b R_1\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,\mathrm{d}x\,\mathrm{d}y=0, \\
&\int_0^a\int_0^b R_2\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,\mathrm{d}x\,\mathrm{d}y=0.
\end{aligned}
\tag{28}
$$

The determinant of the system of Eq. (28) for the coefficients E_{mn} and F_{mn} is set to zero, which yields

$$
\Phi_{\mathrm{m}}=\frac{(1-\nu_0)\pi^2 D}{b^2}\left(\left(\frac{mb}{a}\right)^2+n^2+(I_{mn})^{1/3}\right), \tag{29}
$$

where

$$
I_{mn}=\frac{1024E_1m^2n^2\mu^2h^2}{9D\pi^4\left[\left(\frac{mb}{a}\right)^2+n^2\right]}\left(\frac{b}{a}\right)^4. \tag{30}
$$

Note that these equations are derived for odd values of m and n. If either value of m or n is even, the system of Eq. (28) results into the trivial solutions for E_{mn} and F_{mn}, which is unacceptable. The value of ΔT is obtained using Eq. (4) as

$$
\Delta T=C\left(\left(\frac{mb}{a}\right)^2+n^2+(I_{mn})^{1/3}\right) \tag{31}
$$

where

$$
C=\frac{(1-\nu_0)\pi^2D}{b^2h\left[E_{\mathrm{m}}\alpha_{\mathrm{m}}+(E_{\mathrm{m}}\alpha_{\mathrm{cm}}+E_{\mathrm{cm}}\alpha_{\mathrm{m}})\frac{1}{k+1}+\frac{E_{\mathrm{cm}}\alpha_{\mathrm{cm}}}{2k+1}\right]}. \tag{32}
$$

Here, ΔT_{cr} is the smallest value of ΔT that is obtained when $m=1$ and $n=1$. Therefore

$$
\Delta T_{\mathrm{cr}}=C\left(\left(\frac{b}{a}\right)^2+1+(I_{11})^{1/3}\right), \tag{33}
$$

where I_{11}, using Eq. (30), is defined as

$$I_{11} = \frac{1024 E_1 \mu^2 h^2}{9 D \pi^4 \left[\left(\frac{b}{a}\right)^2 + 1\right]} \left(\frac{b}{a}\right)^4. \tag{34}$$

Setting $\mu = 0$, Eq. (33) is reduced to the critical buckling temperature change of a functionally graded perfect plate, which has been reported by Javaheri and Eslami (2002a) as

$$\Delta T_{\rm cr} = C\left(\left(\frac{b}{a}\right)^2 + 1\right). \tag{35}$$

Also, by setting the power law index equal to zero ($k = 0$), Eq. (33) is reduced to $\Delta T_{\rm cr}$ for homogeneous imperfect plates, which has been reported by Mossavarali et al. (2000) as

$$\Delta T_{\rm cr} = \frac{\pi^2 h^2}{12(1+\nu_0) b^2 \alpha}\left(\left(\frac{b}{a}\right)^2 + 1 + (I_{11})^{1/3}\right). \tag{36}$$

3.2. Buckling of imperfect FGPs under nonlinear temperature change across the thickness

The steady state heat conduction equation and the relative boundary conditions are

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}z}\left(K(z)\frac{\mathrm{d}T}{\mathrm{d}z}\right) = 0, \\ &T = T_{\rm c}, \quad z = \frac{h}{2}, \\ &T = T_{\rm m}, \quad z = -\frac{h}{2}, \end{aligned} \tag{37}$$

where $T_{\rm c}$ and $T_{\rm m}$ are the temperatures of ceramic-rich and metal-rich surfaces, respectively. Using Eq. (1), the heat conduction equation becomes a function of z. The solution is achieved as

$$T(r) = T_{\rm m} + \frac{\Delta T \cdot r \sum_{n=0}^{n=\infty} \frac{\left(-\frac{r^k K_{\rm cm}}{K_{\rm m}}\right)^n}{nk+1}}{\sum_{n=0}^{n=\infty} \frac{\left(-\frac{K_{\rm cm}}{K_{\rm m}}\right)^n}{nk+1}} \tag{38}$$

where $\Delta T = T_{\rm c} - T_{\rm m}$ and

$$r = \frac{2z+h}{2h}. \tag{39}$$

Considering the temperature distribution expressed by Eq. (38) and following the same procedure as the previous loading case, $\Delta T_{\rm cr}$ is obtained as

$$\Delta T_{\rm cr} = H\left(\left(\frac{b}{a}\right)^2 + 1 + (I_{11})^{1/3}\right), \tag{40}$$

where I_{11} is defined by Eq. (34) and

$$H = \frac{(1-\nu_0)\pi^2 D \sum_{n=0}^{n=\infty} \frac{\left(-\frac{K_{\rm cm}}{K_{\rm m}}\right)^n}{nk+1}}{b^2 h \sum_{n=0}^{n=\infty} \frac{\left(-\frac{K_{\rm cm}}{K_{\rm m}}\right)^n}{nk+1}\left[\frac{E_{\rm m}\alpha_{\rm m}}{nk+2} + \frac{E_{\rm m}\alpha_{\rm cm} + E_{\rm cm}\alpha_{\rm m}}{(n+1)k+2} + \frac{E_{\rm cm}\alpha_{\rm cm}}{(n+2)k+2}\right]}. \tag{41}$$

Setting $\mu = 0$, Eq. (40) is reduced to $\Delta T_{\rm cr}$ for a perfect FGP. Also, by assuming $k = 0$, $\Delta T_{\rm cr}$ for an imperfect homogeneous plate is obtained, which has been reported before as (Mossavarali et al., 2000)

$$\Delta T_{\rm cr} = \frac{\pi^2 h^2}{6(1+\nu_0)b^2\alpha}\left(\left(\frac{b}{a}\right)^2 + 1 + (I_{11})^{1/3}\right). \tag{42}$$

3.3. Buckling of imperfect FGPs under linear longitudinal temperature change

Assume a linear temperature variation along the x-direction as

$$T(x) = \Delta T\left(\frac{x}{a}\right) + T_0, \quad 0 \leqslant x \leqslant a, \tag{43}$$

where

$$\Delta T = T_a - T_0. \tag{44}$$

Here, T_a and T_0 are the temperatures at $x = a$ and $x = 0$, respectively. Following the same procedure, $\Delta T_{\rm cr}$ is obtained as

$$\Delta T_{\rm cr} = 2C\left(\left(\frac{b}{a}\right)^2 + 1 + (I_{11})^{1/3}\right), \tag{45}$$

where C and I_{11} are expressed by Eqs. (32) and (34), respectively.

4. Result and discussion

The thermal buckling loads of the rectangular imperfect functionally graded plate are obtained in closed form solutions for the assumed thermal loadings and are represented by Eqs. (33), (40) and (45). These equations indicate that the critical buckling temperature change of an imperfect FGP is increased in com-

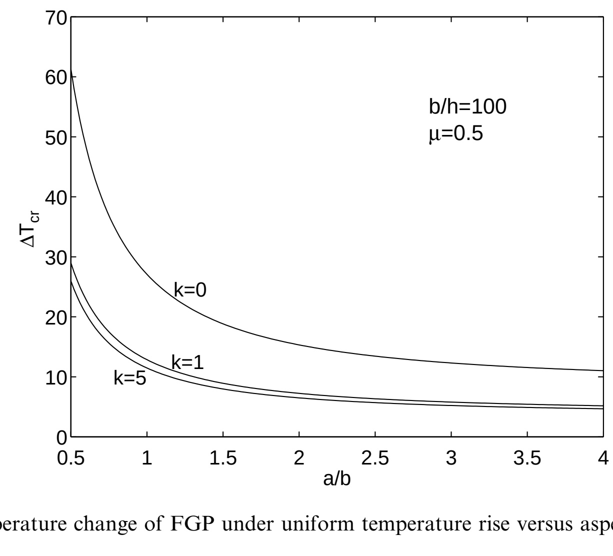

Fig. 1. Critical buckling temperature change of FGP under uniform temperature rise versus aspect ratio and power law index.

parison with a perfect one. The increase in ΔT_{cr} is expressed by an imperfection term $(I_{11})^{1/3}$, which directly depends to the imperfection size μ. Also, investigation of Eq. (34) shows that the imperfection term is affected by the material and geometrical properties of a functionally graded plate. The fact that the thermal buckling load of a plate is increased by existence of geometrical imperfections is fully explained by Murphy and Ferreira (2001). The perfectly flat plate undergoes a symmetric pitchfork bifurcation at the buckling temperature. In contrast, the imperfect plate develops an asymmetric secondary state by means of a saddle-node bifurcation at the higher temperature (Murphy and Ferreira, 2001). The present study confirms this behavior for the functionally graded plates.

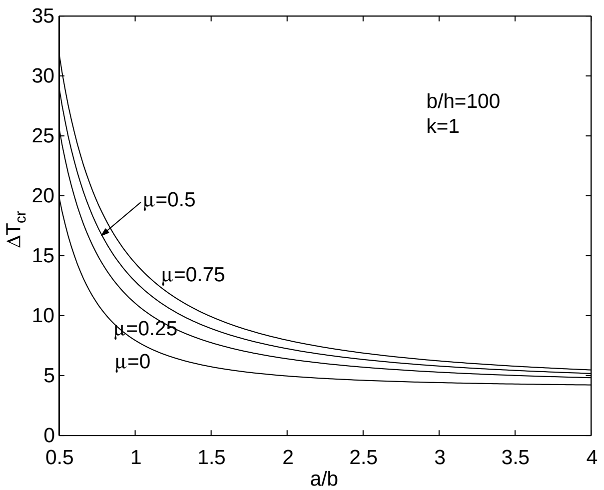

Fig. 2. Critical buckling temperature change of FGP under uniform temperature rise versus aspect ratio and imperfection size.

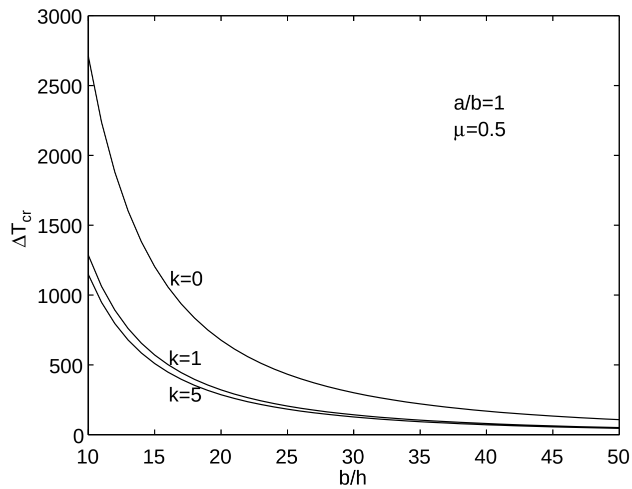

Fig. 3. Critical buckling temperature change of FGP under uniform temperature rise versus b/h and power law index.

As an example, consider an imperfect ceramic-metal FGP that consists of aluminum and alumina with following properties (Javaheri and Eslami, 2002a):

$$\begin{aligned} &E_m = 70 \text{ GPa}, \quad \alpha_m = 23e^{-6}, \quad K_m = 204 \text{ W/mK}, \\ &E_c = 380 \text{ GPa}, \quad \alpha_c = 7.4e^{-6}, \quad K_c = 10.4 \text{ W/mK}, \\ &v_0 = 0.3, \quad h = 0.005 \text{ m}. \end{aligned} \tag{46}$$

The plate is assumed to be fixed simply supported on all its four edges. The variation of ΔT_{cr} versus a/b, b/h, k, and μ are plotted for three types of thermal loadings in Figs. 1–8. Figs. 1, 2, 5, and 8 show that ΔT_{cr}

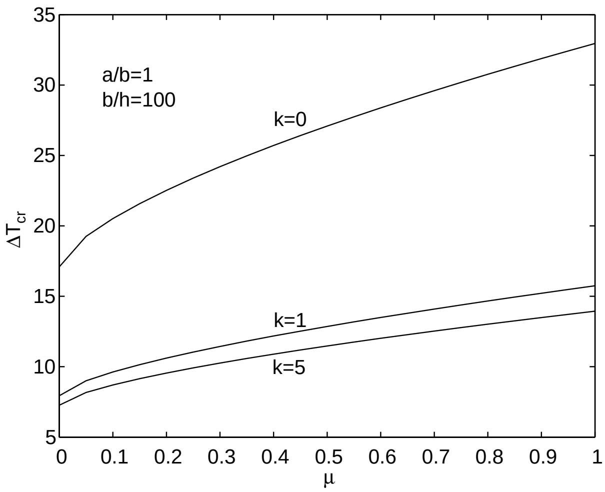

Fig. 4. Critical buckling temperature change of FGP under uniform temperature rise versus imperfection size and power law index.

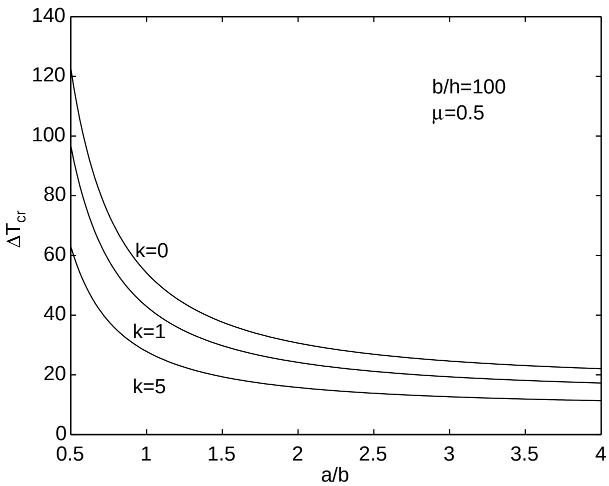

Fig. 5. Critical buckling temperature change of FGP under temperature change across the thickness versus aspect ratio and power law index.

decreases by increasing the aspect ratio a/b. Also, from Figs. 3 and 6 it is seen that ΔT_{cr} decreases by the increase of b/h. Figs. 1, 3–8 reveal that ΔT_{cr} increases by the decrease of power law index k from 5 to 0. In Figs. 1, 3, and 8, it is seen that ΔT_{cr} decreases from $k = 1$ to $k = 5$ to the much less extent than that from $k = 0$ to $k = 1$. Increase in $\Delta\ T_{cr}$ due to enlargement of imperfection size can be easily seen in Figs. 2, 4, and 7. In the case of nonlinear temperature change across the thickness, it is necessary to consider at least the first 20 terms of the series in Eq. (41) in order to achieve ΔT_{cr} with appropriate accuracy.

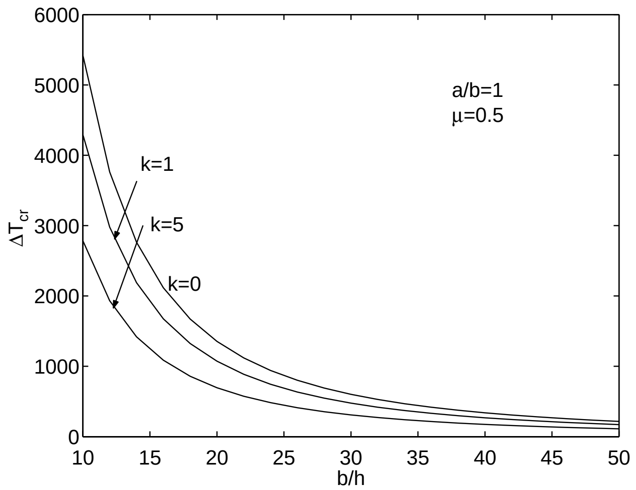

Fig. 6. Critical buckling temperature change of FGP under temperature change across the thickness versus b/h and power law index.

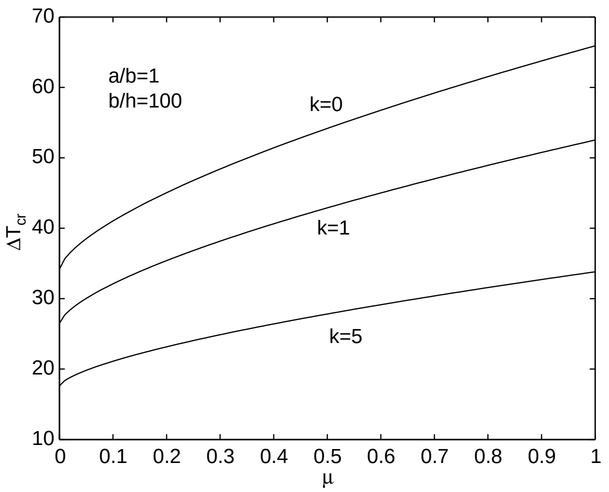

Fig. 7. Critical buckling temperature change of FGP under temperature change across the thickness versus imperfection size and power law index.

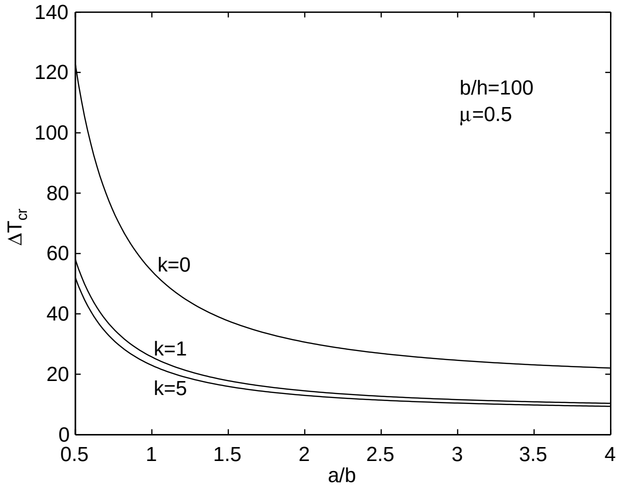

Fig. 8. Critical buckling temperature change of FGP under longitudinal temperature change versus aspect ratio and power law index.

5. Conclusion

In the present paper, equilibrium, stability, and compatibility equations for a rectangular imperfect functionally graded plate are derived. Derivations are based on the classical plate theory and with the assumption of power law composition for the constituent materials. The buckling analysis of such a plate under three types of thermal loadings is investigated. The plate buckles at first mode and the followings are concluded:

1. The critical buckling temperature change, ΔT_{cr}, of an imperfect FGP is greater than a perfect one. This increase is defined by the imperfection term $(I_{11})^{1/3}$, which is a function of imperfection size, geometrical dimensions, mechanical properties of FGP constituents, and the power law index.
2. ΔT_{cr} of a functionally graded plate increases with the increase of imperfection size μ.
3. ΔT_{cr} of an imperfect FGP is reduced when the power law index k increases. The reduction from $k = 0$ to $k = 1$ is considerable. However, for $k > 1$, it is marginal.
4. ΔT_{cr} of an imperfect FGP decreases with the increase of dimension ratios a/b and b/h.
5. ΔT_{cr} of the plate subjected to linear longitudinal temperature change is twice the one subjected to uniform temperature rise.

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